Synchronization and integrability of N-machine system with transfer conductances

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***Abstract*—The tools used for assessment of rotor angle stability should be able to estimate the domain of attraction and the ensuing control laws which would aid in enhancing it. The widely used energy function formulations have limitations in being extended for lossy N-machines owing to the presence of transfer conductances obtained due to the modeling approaches used in transient stability studies. This problem is akin to the N-body problem in celestial mechanics, for which, solutions exist, but finding analytical solutions fail beyond the two body case. The dynamics of the N-body system motivates the authors to assert that integrability and synchronization can exist independently. The major contribution of the paper is the application of the Watanabe-Strotgatz theory for oscillators to the N-machine system and prove that integrability of the system can be achieved only at certain extremes of the transfer conductances, but is generically, not achievable.**

***Index Terms*—constants of motion, energy function, integrability, transfer conductance, Watanabe-Strotgatz theory**

I. INTRODUCTION

THE dynamics of a power system is extremely complex which involve interactions between its three components: generation, transmission and distribution. The interactions between the mechanical power input and the electrical power demand determine the dynamics of the generator which is governed by the swing equation. Solutions of this equation are analyzed to check for any violations in the rotor angles of the generators. The equal area criterion is a tool used for single machine infinite bus structure whereas assessment of the stability of **multi-machine power systems (MMPS)** involves two approaches: the time domain solutions and the Lyapunov based direct methods. The former approach fails to provide sound stability margins and tools for suitable sensitivity analysis [1]. Though this drawback is overcome by the direct methods, constructing good Lyapunov functions(LF) for MMPS has been challenging. Often, one has to use experience or physical insights (e.g., the energy function (EF) for electrical and mechanical systems) to search for an appropriate LF [2]. There have been several papers which have used the concept of EF to assess rotor angle stability of MMPS. Though the EF has served as the LF for two machines, extension to N-machine case is elusive owing to the presence of transfer conductances (TC).

For MMPS, formation of an EF is possible only when TC are negligible. Presence of TC makes the integral of the potential energy term path dependent whereas, formation of EF will be possible only if this integral is path independent. EF formulations are possible only when the dynamical equations governing them are integrable and presence of TC makes the EF non-integrable. The notion of integrability implies that the governing dynamical equations need to be solved in order to achieve global solutions and the most natural procedure to come up with solutions is to find the constants of motion (CoM). A N-dimensional system needs (N-1) CoM for integrability. As it's not feasible to find the required number of CoM for $N > 2$, such systems are non-integrable. A motivating parallel can be drawn from the N-body problem of celestial mechanics. As quoted in [3], solutions to the N-body problem exist. However, analytically solving the dynamical equations is difficult unless integrability helps in.

Apart from assessing stability of system's equilibrium point, the purpose of forming LF is to also find suitable controllers to attain stability, thereby aiding in the enhancement of the domain of attraction. Many control approaches have asserted that the integrability condition needs to be satisfied so as to derive globally convergent stabilizing controllers for MMPS [4]. However, satisfying the integrability condition will remain a holy grail problem for lossy MMPS when the tool to evaluate the stability of its equilibrium points is the EF. Though researchers have attempted to extend the EF formulations for two machines and beyond, the issue of TC hampers generalization to N-machines [5]. Despite the prolonged efforts made by authors to address non-integrability, there has been no analytical proof to show that N-machine systems are non-integrable. This brings in the first contribution of the paper, wherein, using the Watanabe-Strotgatz (WS) theory, non-integrability of N-machine system is proved by considering the entire gamut of dynamics of N-machines.

The inability to account for TC is due to the limitations in modeling of loads which simplifies the formation of the LF. In network reduced models (NRM), the loads are treated as constant impedances and are merged into a network where only generator internal nodes are retained. The load impedances appear as TC in the reduced network which hamper the evaluation of EF beyond two machines. Though the structure preserving models (SPM) include detailed load and generator dynamics, this approach also uses certain assumptions in evaluating the EF [2]. Hence, forming EF for lossy MMPS will remain pervasive. Irrespective of whether EF are integrable or non-integrable, MMPS continue to exist in synchronism. A supporting analogy can be drawn from the N-body problem of celestial mechanics, wherein the celestial bodies continue to exist in synchrony. The second contribution of this work is to reiterate that even though the EF for lossy MMPS is not integrable, N-machines can be synchronized. As proposed in [6], there is no relation between line losses and system instability. **Integrability is not mandatory for synchronization**. Global solutions to such systems are infeasible, however, numerical procedures to assess stability domains exist.

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The existence of synchrony of N-machines despite the presence of TC urges to explore a different approach in understanding the underlying phenomenon. In literature, this has been envisaged by modeling of N-machines as an ensemble of Kuramoto oscillators [7]. The Kuramoto model symbolizes the transient stability problem as a synchronization problem by modeling the machines as phase oscillators. The region of interest for rotor angle stability is the cohesive region, wherein, the oscillators would be phase locked with a region of arc on the unit circle (which corresponds to relative rotor angles $< 180^0$). The model assesses the stability of N-machine with TC and explains that there exists a critical value K_{cr} of the coupling co-efficient K below which the system remains in asynchrony and as $K \rightarrow 1$, oscillators are phase locked. The value of K_{cr} is proportional to the spread of the phase angles of the oscillators and is dependent on the value of TC. Their relationship is explored to show the effect of using a conservative approach in forming a LF and its impact on the domain of attraction which brings in the third contribution of the paper.

The paper is organized as follows: Section II reviews the modeling issues in assessment of transient stability of MMPS using EF as a tool. A brief note on integrability is presented. Section III revisits the synchronization approach of the Kuramoto model and analyzes the effect of TC on K_{cr}. Section IV describes the behavioural properties of Kuramoto model and describes the general procedure for forming CoM for N oscillators with the help of WS theory, thereby, illustrating the regions of integrability. Section V summarizes the discussion and proposes the conclusive remarks and future scope.

II. Transient stability assessment: Modeling issues

The assessment of rotor angle stability is extremely crucial to measure the effect of large disturbances on the system. In a power network comprising of n generator nodes, and m load nodes, for transient stability assessment, rotor dynamics of generators need to be monitored, which can be represented by swing equations of generator i as:

$$M_i\ddot{\delta}_i = P_{mi} - E_i^2 G_{ii} - D_i\dot{\delta}_i - P_{ei}, \quad for\ i \in \{1, \cdots, n\} \quad (1)$$

where, internal voltage of generator $E_i > 0$, mechanical power input $P_{mi} > 0$, inertia $M_i > 0$, damping constant $D_i > 0$. The electrical power output P_{ei} is given as:

$$\begin{aligned} P_{ei} &= \sum_{j=1}^{n} |E_i||E_j|[Re(Y_{ij})cos(\delta_i - \delta_j) + Im(Y_{ij})[sin(\delta_i - \delta_j)] \\ &= \sum_{j=1}^{n} |E_i||E_j||Y_{ij}|sin(\delta_i - \delta_j) + \psi_{ij}] \quad (2) \end{aligned}$$

$$where\ |Y_{ij}| = \sqrt{G_{ij}^2 + B_{ij}^2}, \ \ \psi_{ij} = arctan(G_{ij}/B_{ij}) \quad (3)$$

For SPM, P_{ei} would represent the power flow in the transmission lines having admittance of $G_{ij} + jB_{ij}$. For NPM, Kron reduction is employed wherein only generator nodes are retained and all the load nodes are merged in the reduced admittance matrix, $Y_{red}(orY_{ij})$. G_{ij} represents the TC between generator i and generator j. If $G_{ij} = 0, (\psi_{ij} = 0)$, then the line is treated as lossless, else it is a lossy line. P_{ei} would then represent the power flow exchanged between the machines which is termed as generator flows (GF). Owing to the different modeling techniques of the loads, evaluation of P_{ei} serves as a hindrance in formation of EF.

In NRM, the load impedances appear as ψ_{ij} which account for the TC between the generators. The major difficulty in the analysis of systems with TC is that a closed form expression for the total system energy cannot be obtained. The SPM incorporates detailed dynamic models of loads, yet path dependent integrals are neglected in evaluation of the EF. Though, EF is a powerful tool in analyzing stability, the inherent modeling of power system in evaluating the first swing poses a problem in their evaluation as lossy systems are not integrable. Formulating an EF or Hamiltonian with TC for a N-machine is an un-tractable problem similar to the N-body problem owing to its non-integrability. The concept of integrability and its relation to synchronization is explained in the next section.

A. Integrability and Synchronization

A separate section on integrability is implored to address the obsession of control community to the idea of existence of integrability for formation of suitable control law in order to achieve stability. Given a dynamical system governed by

$$\dot{X} = f(X) \quad (4)$$

with an initial condition $X(0)$, integraility implies solving these equations to obtain the state of the system at time t, i.e. $X(t)$ given $X(0)$. This indicates that integrability involves finding global solutions. According to Poincare, integrating a differential equation is finding a finite expression for the general solution, possibly multivalued, in a finite number of functions [8]. The word finite indicates that integrability is related to a global rather than local knowledge of the solution. Integrability is the property of equations for which all local or global solutions can be obtained either explicitly from the solutions or implicitly from the CoM [9]. Since an EF approach is generally followed to assess stability, a N-body system needs $(N-1)$ CoM for integrability. With dissipative nature of systems, solving dynamical equations analytically is a challenge beyond the two body problem as its difficult to find the required number of CoM for $N > 2$. Hence, its said that such systems are non-integrable. The authors propose to reiterate that solutions exist for three machines and beyond but they cannot be found by using EF as a tool.

Integrability is a rare phenomenon, a typical dynamical system is non-integrable [10]. In the context of power systems, integrability is associated with the EF used to assess stability of MMPS. The effect of the presence of TC is to act as a perturbation to the otherwise, conserved system. Analytically procedures of solving this integral for lossy MMPS fail as the integral becomes path dependent and its value would change with change in the fault location. Hence, energy integrals fail to provide global solutions and systems with TC are non-integrable. Nevertheless, in a realistic scenario,

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TC exist in the system and yet, a N-machine system remains synchronized. This raises questions on the relation between synchronization and integrability. If a system is not integrable, does it imply that it is unstable? Probably not, as supported by the discussion above. Maybe, a different modeling approach is needed to understand the underlying phenomena. This is achieved by modeling N-machines as oscillators, an approach, which has widely been used and is briefed in the next section.

III. Machines as Kuramoto oscillators

The dynamics of the rotor of the generator described by the swing equation, gives a balance between the mechanical output of the rotor and the coupling of this output to the other nodes of the grid. The rotational properties of generators help in visualizing them as oscillators whose dynamics are governed by the Kuramoto model. The analogy between the swing equation and the oscillator dynamical equation is derived from the fact that both are a consequence of Newton's second law and are related to the active power flow. Solutions to Kuramoto model results in estimating synchronization of oscillators whereas solutions to swing equations involve determination of rotor angle stability. The motivation to symbolize machines as oscillators is to view phase locking of oscillators as synchronization of the rotor angles. The equivalence between the two dynamical equations has been explored by [7].

Each oscillator oscillates independently at its natural frequency while the coupling tends to synchronize it to all others. When the coupling is weak, oscillators run incoherently whereas for coupling strength beyond a certain threshold, oscillators are synchronized with each other. The governing equations of the coupled Kuramoto oscillator as shown in Fig.1 are:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{N} K_{ij} sin(\theta_i - \theta_j), for\ i \in \{1, \cdots, N\} \quad (5)$$

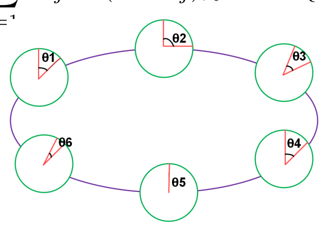


Fig. 1: Kuramoto model of N coupled oscillators

K_{ij} is a matrix comprising of the coupling weights. In order to apply the Kuramoto model in power system parlance, the relationship between the power network model and a first-order model of coupled oscillators is exploited. The singular perturbation analysis is applied to show the congruity between (1) and (5), assuming that the generators are overdamped possibly due to local excitation controllers. Using the relation (6) and (7) to represent the effective power input to generator i and the coupling weights representative of power transferred between generator i and j respectively, (1) would get modified to (8)

$$w_i \equiv (P_{mi} - E_i^2(G_{ii})) \quad (6)$$

$$P_{ij} = |E_i||E_j||Y_{ij}| \quad with \quad P_{ii} = 0 \quad (7)$$

$$M_i\ddot{\delta}_i = -D_i\dot{\delta}_i + \omega_i - \sum_{j=1}^{n} P_{ij} sin(\delta_i - \delta_j + \psi_{ij}) \quad (8)$$

For small inertia over damping ratio ($\frac{M_i}{Di}$) of generators, singular perturbation can be applied to seperate slow and fast dynamics of the system, then (8) reduces to,

$$D_i\dot{\delta}_i = \omega_i - \sum_{j=1}^{n} P_{ij} sin(\delta_i - \delta_j + \psi_{ij}) \quad (9)$$

which captures the power system dynamics sufficiently well during first swing. A corelation between (5) and (9) reveals that with $\psi_{ij} = 0$, they both are alike, where P_{ij} plays the same role as that of the coupling function K_{ij}. The relationship between ψ_{ij} and K_{ij} is described in this section whereas the role of ψ_{ij} in integrability will be discussed in Section IV.

An intuitive representation of synchronization is brought about by the Kuramoto mean field model (KMFM), by taking $K_{ij} = K/N > 0$, resulting in:

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N} sin(\theta_i - \theta_j), for\ i \in \{1, \cdots, N\} \quad (10)$$

(10) can be written in a more convenient form using the order parameter (OP) r, the centroid of the oscillators, which is a natural measure of synchronization and defined as:

$$r(e^{j\phi t}) = \frac{1}{N}\sum_{k=1}^{N} e^{j\theta_k(t)} \quad (11)$$

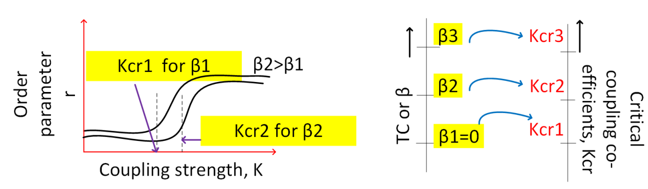
where r measures phase coherence of the oscillators and ϕ measures the average phase. Kuramoto model (10) represented in terms of OP as:

$$\dot{\theta}_i = \omega_i - Krsin(\phi - \theta_i), for\ i \in \{1, \cdots, N\} \quad (12)$$

The equivalent of TC in the KMFM is brought about by adding a term β in the coupling function of (12). It denotes the averaged value of the TC for oscillators.

$$\dot{\theta}_i = \omega_i - Krsin(\phi - \theta_i - \beta), for\ i \in \{1, \cdots, N\} \quad (13)$$

In literature, (13) is also known as the non-uniform Kuramoto model. The effect of change in K on r is depicted in Fig.2a which shows that as β increases, more coupling strength is needed to synchronize oscillators, resulting in an increase in K_{cr} which corresponds in a way, to the stability margin/ the region of attraction of the LF. The effect of increasing β on K_{cr} is shown in Fig.2b. K_{cr1} is the critical coupling strength at $\beta_1 = 0$ and K_{cr2} is the critical coupling strength at β_2 and so on. With an increase in β, K_{cr} also increases.



(a) r vs K (b) Effect of β on K_{cr}

Fig. 2: Relation of r and β with K

The inferences drawn from Fig.2b are:

1) Beyond 2 machine, forming a LF with TC is infeasible, i.e., LF cannot be formed at β_2, β_3....

2) Hence, from a practical point of view, LF is always formed at $\beta_1 = 0$.
3) For e.g., if the system is operating at β_2 and its stability is analyzed at $\beta_1 = 0$, then this would result in an underestimate of the K_{cr}, i.e. the system sees K_{cr1} and not K_{cr2}, resulting in a shrinking in the etimate of domain of attraction.
4) The larger the β, the more would be the error in the estimate of domain of attraction.

This discussion emphasizes that LF can be formed by neglecting the TC but this assumption would be at the cost of sacrificing the estimates of region of attraction.

IV. Behaviour and Integrability of Kuramoto Oscillators

The periodic solutions of (13) fall into three categories: in-phase, splay and incoherent. Fig.3 depicts these states followed by their definitions:

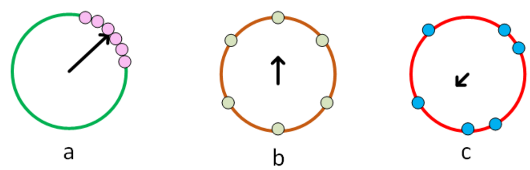


Fig. 3: Periodic solution of Generalized Kuramoto oscillators

1) In-phase state/ synchronous state: A state in which the interaction between the oscillators is attractive and tends to lock them in-phase. If all the oscillators are in the generating mode, then they are said to be phase cohesive.
2) Incoherent/ Asynchronous state: A state in which the interaction between the oscillators is repulsive and tends to lock them in anti-phase
3) Splay state: A state which is the most stable of the asynchronous state.

Fig.3a shows N oscillators which are synchronized and are in the synchronous region. Fig.3b depicts the splay (outspread) states which are the most stable of the incoherent states, Fig.3c. The significance of these states is highlighted as:

- The whole in-phase state is not conducive to synchronization. The oscillators would exist in the generating mode only above K_{cr} which is the phase cohesive region.
- As TC increases, the oscillators start falling apart and spread over the unit circle, well within the phase cohesive region. If TC increases beyond K_{cr}, then the oscillators start spreading out and move towards the incoherent state.
- As the application is to assess rotor angle stability, the dynamics of the oscillators when they are in the synchronous state have only been examined. Though splay states have been discussed in literature, as they are not relevant from machine stability point of view, they have not been explored.
- However, the dynamics at the splay states give lot of insights into the integrability of N-machine systems as proved by the WS theory.

Exploring the whole spectrum of the solutions of an ensemble of N oscillators gives rise to a varied dynamics which help in exposing the domains of integrability. The WS theory aids in providing a full dynamical description of the Kuramoto model of identical oscillators by reducing the dynamics to that of three macroscopic constants along with CoM. As the notion of integrability involves finding CoM, the WS theory finds (N-3) CoM for an N-dimensional system, thereby reducing the dynamical equations to three dimension. This would aid in evaluating the system analytically. Forming a LF, L, using these reduced co-ordinates aids in deciding the stability of the coherent and the incoherent regions. It can be observed that (13) can be written in the generalized form [11]:

$$\dot{\theta}_i = g(t)cos\theta_i + h(t)sin\theta_i \tag{14}$$

With respect to (13), the functions $g(t)$ and $h(t)$ in (14) are $g(t) = Krsin(\phi + \beta)$ and $h(t) = -Krcos(\phi + \beta)$. The ensemble of (14) can be reduced to three time dependent variables γ, Φ and Ψ and $(N-3)$ constants:

$$\begin{aligned} \dot{\gamma} &= -(1-\gamma^2)(g(t)sin\Phi - h(t)cos\Phi) \\ \gamma\dot{\Psi} &= -\sqrt{((1-\gamma^2)))}(g(t)cos\Phi + h(t)sin\Phi) \\ \gamma\dot{\Phi} &= (g(t)cos\Phi - h(t)sin\Phi) \end{aligned} \tag{15}$$

where Ψ and Φ are global variables, and $0 \leq \gamma \leq 1$ is the amplitude of the harmonic force. The reduction of N-dimensional system to three dimensional is achieved using the transformation:

$$\theta_i = \Phi(t) + 2arctan\Big[\frac{\sqrt{(1+\gamma(t))}}{\sqrt{(1-\gamma(t))}}tan[\frac{\varphi_k - \Psi(t)}{2}]\Big] \quad i \in \{1, \cdots, N\} \tag{16}$$

where φ_k are constants. WS have demonstrated that the set of constants φ_k together with the solutions of (15) yields a solution of (14) via transformation (16). Obtaining the initial values of γ, Ψ and Φ and the constants φ_k from the initial conditions θ_i are explained in [11]. Two additional constraints in the incoherent state are imposed on the constants φ_k :

$$\sum_{k=1}^{N} cos\varphi_k = \sum_{k=1}^{N} sin\varphi_k = 0 \tag{17}$$

These two constants (17) along with the (N-3) CoM imposed in the incoherent manifold make the splay state integrable. The graphical representation of (16) is shown in Fig.4 [12]. The reduced co-ordinate variables γ and Φ are analogous to the amplitude of the mean field r and its average phase ϕ [12]. It can be seen from Fig.4 that $\gamma = 0$ corresponds to the incoherent manifold where the phase of the oscillator is equal to the mean field. Note that as γ increases from 0 to 1, the phases of the oscillators move from asynchronous to synchronous regime. A similar phenomena is also observed in the cascade failures in power grid. The line flows under a healthy grid condition follow a Gaussian distribution (GD) which is an assumption. Owing to distrurbances, as lines get overloaded and tripped, the cumulative distribution of line flows collapses from Gaussian to non-Gaussian (NG) distribution which indicates decrease in coherence or movement from the synchronous to the asynchronous mode. Such phenomenon has been discussed in [13] wherein, the GD of line flows is based on the assumptions that the line flows follow the central limit theorem. The analysis of the reduced coordinates of the original system using the WS theory rather, justifies this assumption.

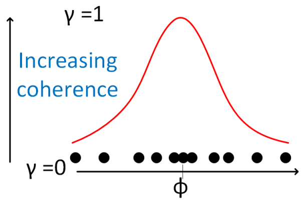


Fig. 4: Variation in coherence with change in γ and Φ

For the analysis of (15) it is convenient to introduce quantities S and T

$$S(\gamma, \Phi) = rsin(\phi - \Phi),\ T(\gamma, \Phi) = -rcos(\phi - \Phi) \quad (18)$$

which gives $r^2 = S^2 + T^2$. They can also be represented via derivatives of a LF $L(\gamma, \Phi)$ in polar coordinates (γ and Ψ):

$$L(\gamma, \Phi) = \frac{1}{N}\sum_{k=1}^{N} log\Big(\frac{1 - \gamma cos(\varphi - \Psi)}{\sqrt{(1-\gamma^2)}}\Big),$$

$$as\ T = (1-\gamma^2)\frac{\partial L}{\partial \gamma};\ S = -\frac{\sqrt{(1-\gamma^2)}}{\gamma}\frac{\partial L}{\partial \Psi} \quad (19)$$

The time derivative of L is given by:

$$\dot{L} = RKr^2cos\beta \quad (20)$$

Depending on the sign of $\dot{L}$, the stability of the reduced system can be analyzed. For the case of linear coupling, R, K and β are constant, then (20) implies that if:

1) $cos\beta < 0 \Rightarrow \beta > \pi/2$, L decreases, $0 < \gamma << 1$ and an incoherent state with zero mean field $r = 0$ sets in.
2) $cos\beta > 0 \Rightarrow \beta < \pi/2$, L grows and a fully synchronous state with $\gamma \to 1$ establishes.
3) $cos\beta = 0$, $\beta = \pi/2$, splay state sets in which is the most stable of the incoherent states. This corresponds to a full lossy case where the system is integrable.
4) L has a minimum $L = 0$ at the origin $\gamma = 0$, and tends to infinity on the unit circle $\gamma = 1$.

Fig.5 shows the regions at which synchronous, asynchronous and integrable behaviour can be obtained. It illustrates the transition from asynchrony to synchrony by virtue of variation in γ. The repulsive region comprises of two modes: the chaotic incoherent state and the stable splay state. As $\gamma \approx 1$ and the coupling increases beyond K_{cr}, frequency synchronization and phase co-hesiveness of the oscillators is achieved. At the border between attraction and repulsion $cos(\beta) = 0$ (assuming $K \neq 0$) the system becomes integrable.

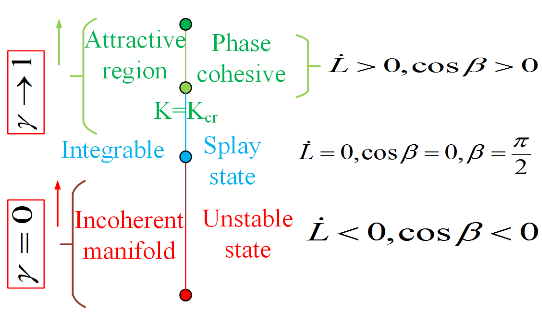


Fig. 5: Stability of the incoherent manifold

V. Conclusions and Future Scope

The focus of this work is to elucidate the limitations in the assessment of rotor angle stability of lossy MMPS with EF formulations as a tool. Global solutions to such systems are infeasible, however, numerical procedures to assess stability domains exist. The discussions help in putting to rest the notion of finding integrable energy functions for lossy MMPS. The formation of LF and thereby the control action by neglecting the TC would result in a shrinking in the region of attraction. The whole endeavour is to draw parallel between the stability of the N-body problem and the lossy MMPS, which reveals that lossy MMPS continue to remain in equilibrium irrespective of the presence of TC. The WS theory provides a concrete proof that lossy MMPS are non-integrable and efforts to find energy function for such systems would be unproductive.

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References

[1] M.Pavella, D.Ernst, and D. Vega. *Transient stability of power systems: a unified approach to assessment and control*, Springer Science & Business Media, 2000, volume 581.
[2] K.R. Padiyar, *Structure Preserving Energy Functions in Power Systems Theory and application* CRC Press, Taylor and Francis group, FL, 2013.
[3] F. Diacu, "*The solution of the N-body problem*", The Mathematical Intelligence, Vol18, No3, 1996.
[4] R. Ortega, M. Galaz, A. Astolfi, Y.Sun, and T. Shen, "*Transient Stabilization of Multimachine Power Systems With Nontrivial Transfer Conductances*", IEEE Trans on Automatic Control, Vol. 50, No. 1, Jan 2005.
[5] Hsiao-Dong Chiang, *Direct Methods for Stability Analysis of Electric Power Systems:Theoretical Foundation, BCU Methodologies, and Applications*, John Wiley & sons, NJ, 2011.
[6] N. Narasimhamurti, "*On the Existence of Energy Function for Power Systems with Transmission Losses*", IEEE Trans on Circuits and Systems, Vol. CAS-31, No. 2, Feb1984.
[7] F. Dorfler, F. Bullo, "*Synchronization and Transient stability in power networks and non-uniform Kuramoto Oscillators*", SIAM J. Control Optimization, Vol. 50, No. 3, pp. 16161642, 2012.
[8] H. Poincare, "*Les methodes nouvelles de la Mecanique Celeste*", Gauthier Villars, Paris (1892)
[9] A. Goriely, *Integrability and non-integrability of non-linear systems*, Advanced series in non-linear dynamics, Vol 19.
[10] B. Grammaticos and A. Ramani, *Integrability −and How to Detect It*, Lect. Notes Phys. 638, 31 − 94(2004)
[11] Shinya Watanabe, Steven H. Strogatz, "*Constants of motion for superconducting Josephson arrays*", Physica D 74 (1994) 197− 253.
[12] A. Pikovsky, M. Rosenblum,"*Self-organized partially synchronous dynamics in populations of nonlinearly coupled oscillators*", Physica D 238 (2009) 2737
[13] S.R. Gupta, R. Kambli, S.R. Wagh and F.S. Kazi,"*Support-Vector-Machine-Based Proactive Cascade Prediction in Smart Grid Using Probabilistic Framework*", IEEE Trans on Industrial Electronics, Vol. 62, No. 4, April 2015.